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Estimation of Elastic Coefficients in Beams with Parameter Constraints: Penalization*

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The estimation of elastic parameters in beams in the presence of pointwise and norm constraints is considered. It is shown that these constraints tend to regularize the optimal parameter. This additional smoothness is useful in analyzing the limit behavior of penalized finite-dimensional approximating problems. Numerical results are presented. © 1988 Academic Press, Inc.

0. INTRODUCTION

In this note we consider estimation of the elastic parameter a in a time independent model for the transverse deformation of a beam

$$(a(x)\, u_{xx})_{xx} - (a_1(x)\, u_x)_x + a_2(x)\, u = f \qquad \text{in } \Omega, \tag{0.1}$$

where $\Omega = (0, 1)$. We study clamped boundary conditions

$$u(0) = u_x(0) = u(1) = u_x(1) = 0 \tag{0.2}$$

although other boundary conditions may be treated using the techniques in this paper. In particular, given an observation $z = z(x)$ or $z = \{z_i\}_{i=1}^{\omega}$, we seek $\bar{a} \in Q_{\text{ad}}$, the set of admissible coefficients, such that

$$J(a_0) = \min\{J(a) : a \in Q_{\text{ad}}\}, \tag{0.3}$$

where $J(a)$ represents a fit-to-data functional. In this work we consider the functionals

$$J_1(a) = \int_0^1 (u(x; a) - z(x))^2\, dx \tag{0.4}$$

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and

$$J_2(a) = \sum_{i=1}^{\omega} (u(x_i; a) - z_i)^2. \tag{0.5}$$

The set of admissible parameters Q_{ad} should be constructed such that

(i) there exists a unique solution to (0.1) and (0.2) for each $a \in Q_{\mathrm{ad}}$,

(ii) there is sufficient compactness to assure the existence of a solution to (0.3).

We take

$$Q_{\mathrm{ad}} = \{a \in H^1(\Omega) : a(x) \geqslant \nu > 0 \quad \text{and} \quad \|a\|_{H^1(\Omega)} \leqslant k\}.$$

The above problem has received attention for time-dependent models [2, 3, 7]. There the problem is to estimate a by a history matching technique several observations at different times. In the present case there is no such information available. Moreover, in the literature for this class of problems very little attention has been given to the role of the constraints; see, however, [6]. In Section 2 we show (cf. [6]) that the constraints tend to regularize the optimal coefficient $\bar{a}$. These regularity properties allow us to translate the constraints for the infinite-dimensional prroblem to the finite-dimensional problems. We consider the effect of the regularity on penalized problems in Section 3. Finally, in Section 4 we indicate some results of numerical experiments.

1. Preliminaries

We study the model equaiton (0.1) with boundary conditions (0.2). The following assumptions are made:

(A1) $a \in H^1(\Omega)$ and $a(x) \geqslant \nu > 0$ in Ω;

(A2) $a_1 \in H^1(\Omega)$ and $a_2 \in L^\infty(\Omega)$ with $a_1(x) \geqslant 0$ in Ω and $a_2(x) \geqslant 0$ a.e. in Ω;

(A3) $f \in L^2(\Omega)$;

(A4) $f \in H^{-2}(\Omega)$.

In this paper we concentrate on estimating the parameter a. Hence, we define the set

$$Q = \{a \in H^1(\Omega) : \text{(A1) holds}\}. \tag{1.1}$$

It is well known that we may associate with (0.1)–(0.2) a bilinear form

$$L(u, v) = \int_\Omega (a(x)\, u_{xx} v_{xx} + a_1(x)\, u_x v_x + a_2(x)\, uv)\, dx \tag{1.2}$$

defined on $H_0^2(\Omega)$. A simple computation shows that for $\phi \in H_0^2(\Omega)$,

$$L(\phi, \phi) \geqslant \gamma \|\phi\|_{H^2}^2. \tag{1.3}$$

where $\gamma = v/7$.

Moreover, we see that for ϕ, $\psi \in H^2(\Omega)$,

$$|L(\phi, \psi)| \leqslant \beta \|\phi\|_{H^2(\Omega)} \|\psi\|_{H^2(\Omega)},$$

where $\beta = \max(\|a\|_{L^\infty(\Omega)}, \|a_1\|_{L^\infty(\Omega)}, \|a_2\|_{L^\infty(\Omega)})$.

Using this bilinear form the problem (0.1), (0.2) may be represented in the weak form as the following.

Find $u = H_0^2(\Omega)$ such that with $f \in H^{-2}(\Omega)$,

$$L(u, \phi) = \langle f, \phi \rangle \tag{1.4}$$

for all $\phi \in H_0^2(\Omega)$. For any $\phi \in H_0^2(\Omega)$ we have

$$|\langle f, \phi \rangle| \leqslant \|f\|_{H^{-2}(\Omega)} \|\phi\|_{H^2(\Omega)}.$$

From the Riesz Representation Theorem we have the following.

THEOREM 1.1. *Let* (A1), (A2), (A4) *hold. There exists a unique solution* $u \in H_0^2(\Omega)$ *of* (1.3). *Further, the solution satisfies*

$$\|u\|_{H^2(\Omega)} \leqslant \frac{1}{\gamma} \|f\|_{H^{-2}(\Omega)}.$$

With assumptions (A1)–(A3) we have in fact a stronger result.

THEOREM 1.2. *If* (A1)–(A3) *hold, then the solution u of* (1.3) *belongs to* $H_0^2(\Omega) \cap H^3(\Omega)$ *and*

$$\|u\|_{H^3(\Omega)} \leqslant C(a) \|f\|_{L^2(\Omega)},$$

where the mapping $a \to C(a)$ *takes bounded sets of* Q *into bounded sets in* $\mathbb{R}^+ \cup \{0\}$.

Proof. See [12].

The following result will be useful in Section 2.

THEOREM 1.3. *Let* (A1) *and* (A2) *hold and let* $f \in H^{-(1/2+\varepsilon)}(\Omega)$ *for* $\varepsilon > 0$. *Then* $u \in H_0^2(\Omega) \cap H^{(11/4-\varepsilon/2)}(\Omega)$ *and*

$$\|u\|_{H^{(11/4-\varepsilon/2)}(\Omega)} \leqslant \bar{C}(a) \|f\|_{H^{-(1/2+\varepsilon)}(\Omega)},$$

where $a \to C(a)$ is a mapping that takes bounded sets in Q to bounded sets in $\mathbb{R}^+$.

Proof. We obtain this result by interpolation in which we define an operator T by $Tf = u$, where u satisfies (1.3). In this case we see that

$$\| Tf \|_{H^2(\Omega)} \leqslant \frac{1}{\gamma} \| f \|_{H^{-2}(\Omega)}$$

from Theorem 1.1 and

$$\| Tf \|_{H^3(\Omega)} \leqslant C(a) \| f \|_{H^0(\Omega)}$$

from Theorem 1.2. From interpolation theory [4, 8, 11], we conclude that for $\varepsilon > 0$,

$$u \in H_0^2(\Omega) \cap H^{(11/4 - \varepsilon/2)}(\Omega).$$

Moreover, we have

$$\| Tf \|_{11/4 - \varepsilon/2} \leqslant \bar{C}(a) \| f \|_{-(1/2 + \varepsilon)},$$

where $\bar{C}(a) = (1/\gamma)^{3/4 - \varepsilon/2}\, C(a)^{1/4 + \varepsilon/2}$.

We now give results concerning the continuity properties of the mapping $a \to u(a)$, where we denote the dependence of u on a by $u(a)$.

THEOREM 1.4. *Let $a^{(k)}$ satisfy* (A1) *for each k, let* (A2)–(A3) *hold, and suppose that $a^{(k)} \to a$ converges weakly in $H^1(\Omega)$ as $f^{(k)} \to f$ in $H^{-(1/2+\varepsilon)}(\Omega)$ for $e \in (0, \frac{3}{2})$. Then $u^{(k)} \to u$ strongly in $H^2(\Omega)$.*

Proof. Since $a^{(k)} \to a$ weakly in $H^1(\Omega)$, it is bounded. Thus, from Theorem 1.3 it follows that the sequence of solutions $\{u^{(k)}\}$ is bounded in $H^{(11/4 - \varepsilon/2)}(\Omega)$. Hence, there exists a subsequence $\{a^{(k_i)}\}$ such that

$$\begin{aligned} a^{(k_i)} &\to a && \text{in } L^\infty(\Omega) \\ u^{k_i} &\to \tilde{u} && \text{in } H^2(\Omega). \end{aligned}$$

It follows that a satisfies (A1). By considering the equations for any $\phi \in H_0^2(\Omega)$,

$$\int_\Omega (a^{(k_i)} u_{xx}^{(k_j)} \phi_{xx} + a_1 u_x^{(k_i)} \phi_x + a_2 u^{(k_i)} \phi)\, dx = \int_\Omega f_{k_i} \phi\, dx,$$

we see that in the limit

$$\int_\Omega (a \tilde{u}_{xx} \phi_{xx} + a_1 \tilde{u}_x \phi_x + a_2 \tilde{u} \phi)\, dx = \int f \phi\, dx.$$

By uniqueness we see that $\tilde{u} = u$ and $u^{(k)} \to u$. The result follows from a subsequence argument and uniqueness of $u(a)$. ∎

To approximate the solution of (1.3) we introduce a finite-dimensional subspace S^N of $H_0^2(\Omega)$ along with a bounded linear operator $\mathscr{I}^N: H_0^2(\Omega) \to S^N$ with the properties

$$\begin{aligned} &\phi \in H_0^2(\Omega) \Rightarrow \|\phi - \mathscr{I}^N\phi\|_{H^2(\Omega)} \leqslant C\,\|\phi\|_{H^2(\Omega)} \\ &\phi \in H_0^2(\Omega) \cap H^4(\Omega) \Rightarrow \|\phi - \mathscr{I}^N\phi\|_{H^2(\Omega)} \leqslant \delta(N)\|\phi\|_{H^4(\Omega)}, \end{aligned} \tag{H1}$$

where $\delta(N) \to 0$ as $N \to \infty$.

Let $u^N \in S^N$ be the solution of the finite-dimensional problem: find $u^N \in S^N$ such that

$$L(u^N, v) = (f, v) \tag{1.5}$$

for any $v \in S^N$. From Cea's lemma [5] we see that

$$\begin{aligned} \|u - u^N\|_{H^2(\Omega)} &\leqslant \inf_{\phi \in S^N} \|u - \phi\|_{H^2(\Omega)} \\ &\leqslant \|u - \mathscr{I}_u^N\|_{H^2(\Omega)}. \end{aligned}$$

From (H1), Theorem 1.2, and the theory of the interpolation of linear operators [4, 11], we conclude that

$$\|u - u^N\|_{H^2(\Omega)} \leqslant C^{1/2}(\delta(N))^{1/2}\, C(a)\|f\|_{L^2(\Omega)}. \tag{1.6}$$

THEOREM 1.5. *Assume* (A1)–(A3) *and* (H1). *Then inequality* (1.5) *provides a rate of convergence of the solution of* (1.5) *to u that is uniform for the coefficient a in bounded subsets of Q.*

Let us now formulate the estimation problem that we consider. Let (A1)–(A3) hold. Define the set

$$Q_{\text{ad}} = \{a \in Q : \|a\|_{H^1(\Omega)} \leqslant K\}. \tag{1.7}$$

We assume that $K > \nu/\sqrt{2}$ so that $Q_{\text{ad}} \neq \varnothing$. Observe also that Q_{ad} is a closed, bounded, convex subset of $H^1(\Omega)$. In this work we study the particular fit-to-data functionals

$$J_1(a) = \int_\Omega (u(a) - z)^2\, dx \tag{1.8}$$

and

$$J_2(a) = \sum_{i=1}^{\omega} (u(x_i, a) - z_i)^2. \tag{1.9}$$

The estimation problem is stated as an optimization problem:

$$\text{Find } a^{(i)} \in Q_{\text{ad}} \text{ such that } J_i(a^{(i)}) = \text{infimum } \{ J_i(a) : a \in Q_{\text{as}} \}. \quad (1.10)$$

The existence of a solution to (1.10) follows from the sequential weak compactness of Q_{ad} and Theorem 1.4.

We next wish to show that the mappings $a \to J_i(a)$, $i = 1, 2$, of $H^1(\Omega)$ into $\mathbb{R}$ are Fréchet differentiable.

THEOREM 1.6. *The mappings $a \to J_i(a)$, $i = 1, 2$, of $H^1(\Omega)$ into $\mathbb{R}$ are Fréchet differentiable and for $i = 1, 2$,*

$$DJ_i(a) = -2u_{xx}(a)\, P^{(i)}_{xx}(a),$$

where $P^{(i)}(a)$ is the solution of the boundary value problems (1.13).

Proof. We begin by finding the Gâteaux derivative of $J_i(a)$ in the direction of h. Thus, setting $v = \delta u(a, h)$, the variation of u in the direction of h, we have

$$\begin{aligned} \delta J_1(a, h) &= 2 \int_\Omega (u(a) - z)\, v\, dx \\ \delta J_2(a, h) &= 2 \sum_{i=1}^{\omega} (u(x_i, a) - z_i)\, v(x_i)\, dx, \end{aligned} \quad (1.11)$$

where v satisfies

$$\begin{gathered} (av_{xx})_{xx} - (a_1 v_x)_x + a_2 v = -(hu(a)_{xx})_{xx} \\ v(0) = v(1) = v_x(0) = v_x(1) = 0. \end{gathered} \quad (1.12)$$

Introducing the boundary value problems

$$\begin{gathered} (aP^{(1)}_{xx})_{xx} - (a_1 P^{(1)}_x)_x + a_2 P^{(1)} = u(a) - z \\ (aP^{(2)}_{xx})_{xx} - (a_1 P^{(2)}_x)_x + a_2 P^{(2)} = \sum_{k=1}^{\omega} (u(x_k, a) - z_k)\, \delta_{x_k} \\ P^{(i)}(0) = P^{(i)}(1) = P^{(i)}_x(0) = P^{(i)}_x(1) = 0, \qquad i = 1, 2, \end{gathered} \quad (1.13)$$

multiplying (1.12) by $P^{(i)}$, and integrating we may express (1.11) as

$$\delta J_i(a, h) = -2 \int_\Omega u_{xx}(a)\, P^{(i)}_{xx}(a)\, h\, dx. \quad (1.14)$$

Since $u_{xx}(a)$ and $P^{(i)}_{xx}(a)$ are in $L^2(\Omega)$ for each a, the Gâteaux derivative $\delta J_i(a, \cdot)$ belongs to $L(H^1(\Omega), (H^1(\Omega))^*)$ for each $a \in Q$. To verify that J_i is

Fréchet differentiable, we show that $a \to \delta J_i(a, \cdot)$ is continuous from $H^1(\Omega)$ into $L(H^1(\Omega), (H^1(\Omega))^*)$. But this follows from Theorem 1.4, which establishes

(i) $a \to u(a)$ is continuous from $H^1(\Omega)$ into $H_0^2(\Omega)$,

(ii) the imbedding of $H_0^2(\Omega)$ into $C^0(\bar{\Omega})$ is continuous,

(iii) $(a, f) \to P^{(i)}(a, f)$ is continuous from $H^1(\Omega) \times H^{-(1/2+\varepsilon)}(\Omega)$ into $H^{11/4-\varepsilon/2}(\Omega)$.

We conclude that the mapping $a \to u_{xx}(a)\, P_{xx}^{(i)}(a)$ is continuous from $H^1(\Omega)$ into $L^2(\Omega)$. Thus, we see that

$$\begin{aligned}
&|\delta J_i(\bar{a}, h) - \delta J_i(a, h)| \\
&\quad \leqslant 2 \int_\Omega |u_{xx}(\bar{a})\, P_{xx}^{(i)}(\bar{a}) - u_{xx}(a)\, P_{xx}^{(i)}(a)|\, |h|\, dx \\
&\quad \leqslant 2\, \|u_{xx}(\bar{a})\, P_{xx}^{(i)}(\bar{a}) - u_{xx}(a)\, P_{xx}^{(i)}(a)\|_{L^2(\Omega)}\, \|h\|_{L^2(\Omega)}.
\end{aligned}$$

We see that J_i is indeed Fréchet differentiable with

$$DJ_i(a) = -2u_{xx}(a)\, P_{xx}^{(i)}(a)$$

for $i = 1, 2$. ∎

2. Regularity of the Optimal Parameter

In this section we establish regularity properties of optimal parameters that are solutions of (1.9). Our results are dependent upon the existence of Lagrange multipliers associated with constraints and are a consequence of a generalized Kuhn–Tucker theorem. Under certain regular point conditions a solution of (1.10) is a critical point of a Lagrangian functional. Hence, the optimal parameter can be shown to solve a Euler equation. It is this property that enables us to deduce the regularity of optimal parameters. We find that the regularity properties of the optimal parameters of (1.10) with fit-to-data functionals (1.8) and (1.9) are the same. Moreover, we find that either the optimal parameter is on the boundary of the contraint set or is a solution of the unconstrained problem.

We begin by specifying several mappings. For $p > \frac{1}{2}$, $H^p(\Omega) \subset C^0(\bar{\Omega})$. Hence we define

$$\begin{aligned}
&G_1 : H^1(\Omega) \to H^p(\Omega) &&\text{by} \quad G_1(a) = v - a \\
&G_2 : H^1(\Omega) \to \mathbb{R} &&\text{by} \quad G_2(a) = \|a\|_{H^1(\Omega)} - k^2 \qquad (2.1) \\
&G : H^1(\Omega) \to H^p(\Omega) \times \mathbb{R} &&\text{by} \quad G(a) = (G_1(a), G_2(a)).
\end{aligned}$$

With these functionals we formulate the minimization problem as

$$\begin{aligned} &\text{minimize } J_i(a) \\ &\text{subject to } a \in H^1(\Omega),\ G_1(a) \leqslant 0,\ G_2(a) \leqslant 0. \end{aligned} \tag{2.2}$$

Certainly, the mappings G_1 and G_2 are continuously Fréchet differentiable with

$$\begin{aligned} DG_1(a)\,h &= -h \\ DG_2(a)\,h &= 2(a, h)_{H^1(\Omega)}. \end{aligned} \tag{2.3}$$

To apply the Kuhn–Tucker theorem it is necessary to establish that solutions of (2.2) satisfy certain regular point conditions [13].

DEFINITION 2.1. An admissible element a is a regular point of the constraints $G_1(a) \leqslant 0$, $G_2(a) \leqslant 0$ if

$$(0, 0) \in \operatorname{int}\{G(a) + \text{range } DG(a)\,h + (H^p(\Omega))_+ \times \mathbb{R}_+\}, \tag{2.4}$$

where $(H^p(\Omega))_+ = \{\phi \in H^p(\Omega) : \phi(x) \geqslant 0\}$.

Remark 2.1. Note that

$$(\psi, r) \in (G(a) + \text{range } DG(a)\,h) + ((H^p(\Omega))_+ \times \mathbb{R}_+)$$

implies that there exist a and h in $H^1(\Omega)$, $\phi \in (H^p(\Omega))_+$, and $r_+ \geqslant 0$ such that

$$\psi = v - a - h + \phi \tag{2.5}$$

and

$$r = \|a\|^2_{H^1(\Omega)} - k^2 + 2(a, h)_{H^1(\Omega)} + r_+. \tag{2.6}$$

LEMMA 2.1. *Every admissible parameter $a \in Q_{\text{ad}}$ is a regular point of the constraints $G_1(a) \leqslant 0$ and $G_2(a) \leqslant 0$.*

Proof. We show that every (ϕ, r) with $\phi \in H^p(\Omega)$, $p > \frac{1}{2}$, is contained in the set in (2.4) if

$$|r|_R + \|\phi\|_{H^p(\Omega)} < \varepsilon = \min\left(\frac{1}{4C\,\|a\|_{H^1(\Omega)}}, \frac{1}{4}\right)(k^2 - v^2), \tag{2.7}$$

where $|\phi|_{C^0(\bar\Omega)} \leqslant C\,\|\phi\|_{H^p(\Omega)}$.

Let $\Phi(x) = \phi(x) - \min_{x \in \Omega} \phi(x)$ and note that $\Phi \in (H^p(\Omega))_+$. Set $h = v - a - \min \phi$ so that $h \in H^1(\Omega)$. Then the element $\phi = v - a - h + \Phi$ is of the form given in (2.5).

Next we must find $p \in \mathbb{R}_+$ such that

$$r = \| a \|^2_{H^1(\Omega)} - k^2 + 2(a, h)_{H^1(\Omega)} + p. \tag{2.8}$$

This is equivalent to

$$\begin{aligned} r = \| a \|^2_{H^1(\Omega)} - k^2 + 2((a, v)_{L^2(\Omega)} - \| a \|^2_{H^1(\Omega)} \\ - (a, \min \phi)_{L^2(\Omega)}) + p. \end{aligned}$$

However, we see that

$$\begin{aligned} &\| a \|^2_{H^1(\Omega)} - k^2 + 2(a, v)_{L^2(\Omega)} - 2 \| a \|^2_{H^1(\Omega)} - 2(a, \min \phi)_{L^2(\Omega)} \\ &\leqslant - \| a \|^2_{H^1(\Omega)} - k^2 + \| a \|^2_{L^2(\Omega)} + v^2 + 2 \, |\min \phi| \, \| a \|_{L^2(\Omega)} \\ &\leqslant (v^2 - k^2) + 2C \, \| \phi \|_{H^p(\Omega)} \, \| a \|_{L^2(\Omega)}. \end{aligned}$$

Now using inequality (2.7), we have

$$\begin{aligned} &\leqslant (v^2 - k) + 2C \left(\frac{1}{4C \, \| a \|_{H^1(\Omega)}} (k^2 - v^2) \right) \| a \|_{L^2(\Omega)} \\ &\leqslant \frac{1}{2} (v^2 - k^2). \end{aligned}$$

It follows then that

$$r \leqslant \frac{1}{2} (v^2 - k^2) + p$$

and

$$p \geqslant r + \frac{1}{2} (k^2 - v^2) > 0,$$

verifying the lemma. ∎

It follows from the above lemma [13] that there exist Lagrange multipliers $\lambda_1^* \in (H^p(\Omega))^*_+$, $p > \frac{1}{2}$ and $\lambda_2 \in \mathbb{R}_+$, such that the Lagrangian

$$\Gamma_i(a) = J_i(a) + \langle \lambda_1^*, G_1(a) \rangle + \lambda_2 G_2(a)$$

is stationary at $a^{(i)}$. We note that λ_1^* and λ_2 may depend upon i as well, but for simplicity we suppress the superscript. If we restrict $h \in H_0^p(\Omega)$, $p > \frac{1}{2}$, then we may associate the functional λ_1^* with a distribution $\lambda_1 \in H^{-p}(\Omega)$. That is, we have

$$\langle \lambda_1^*, h \rangle = (\lambda_1, h)$$

for any $h \in H_0^1(\Omega) \subset H_0^p(\Omega)$. We find then that

$$DF_i(a^{(i)})\,h = -\int_\Omega 2u_{xx}P_{xx}^{(i)}\,dx + (\lambda_1, DG_1(a^{(i)})\,h) + \lambda_2 DG_2(a^{(i)})\,h$$
$$= (-2u_{xx}P_{xx}^{(i)} - \lambda_1 + 2\lambda_2(-a_{xx}^{(i)} + a^{(i)}), h).$$

Since for any $h \in H_0^2(\Omega)$ the solution $u(a^{(i)} + \varepsilon h)$ is well-defined for $0 < \varepsilon < (v-\delta)/C \,\|h\|_{H^1(\Omega)}$, where $0 < \delta < v$ and $\|h\|_{C^0(\bar{\Omega})} \leqslant C\,\|h\|_{H^1(\Omega)}$, we see that

$$D\Gamma(a^{(i)})\,h = 0$$

for any $h \in H_0^1(\Omega)$. Moreover, it follows that

$$\lambda_2(-a_{xx}^{(i)} + a^{(i)}) = u_{xx}P_{xx}^{(i)} + \lambda_1/2 \tag{2.9}$$

holds in $H^{-1}(\Omega)$. Now from the regularity results on u and $P^{(i)}$, we see that $u_{xx} \in H^1(\Omega)$ and $P_{xx}^{(i)} \in L^2(\Omega)$ for $i = 1, 2$. Accordingly, the right side of (2.9) belongs to $H^{-p}(\Omega)$. We conclude that if $\lambda_2 \neq 0$, then $a^{(i)}$ belongs to $H^{2-p}(\Omega)$ for $p > \frac{1}{2}$.

LEMMA 2.2. *If* $\lambda_2 \neq 0$, *then any solution* $a^{(i)}$ *of* (2.2) *belongs to* $H^{3/2-\varepsilon}(\Omega)$ *for* $\varepsilon > 0$.

In addition to the above conclusion the Kuhn–Tucker theorem also implies that

$$\langle \lambda_1^*, G_1(a^{(i)}) \rangle = 0 \tag{2.10}$$

$$\lambda_2 G_2(a^{(i)}) = 0. \tag{2.11}$$

We now determine sufficient conditions to assue that $\lambda_2 \neq 0$. We find that indeed $\lambda_2 \neq 0$ unless there exists some special cnditions on some subinterval of Ω. We specify the following assumptions:

(C1) There do not exist a subinterval $I \subset \Omega$ and constants c_1 and c_2 such that $u = z = a_{1x}c_2 + a_2c_1$ on I.

(C2) There do not exist a subinterval $I \subset \Omega$ and constants c_1 and c_2 such that $f = c_1a_2 + c_2(a_{1x} + xa_2)$ on I.

(C3) $a \not\equiv v$.

(C4) There does not exist a subinterval $I \subset \Omega$ such that $a_2(x) = a_{1x}/x + C$.

LEMMA 2.3. *Assume* (C1)–(C3). *Then* $\lambda_2 \neq 0$ *in* (2.9).

Proof. Let $\lambda_1^* \in (H^p(\Omega))_+^*$ and $\lambda_2 \in \mathbb{R}_+$ be Lagrange multipliers associated with the constraints $G_1(a) \leqslant 0$ and $G_2(a) \leqslant 0$. Assume $\lambda_2 = 0$.

Now since $(H^p(\Omega))^*_+ \subset (H^1(\Omega))^*_+$, $\lambda_2^* \in (H^1(\Omega))^*_+$. Hence, we have for $h \in H^1(\Omega)$,

$$-2\int_\Omega u_{xx} P^{(i)}_{xx} h\,dx - \langle \lambda_1^*, h\rangle = 0. \tag{2.12}$$

We use the following characterization of elements in $(H^1(\Omega))^*$ (cf. [1]). If $\lambda_1^* \in (H^1(\Omega))^*$, then there exists $\lambda_1 \in H^{-1}(\Omega)$ and real numbers α and β such that

$$\begin{aligned}\langle \lambda_1^*, h\rangle &= (\lambda_1, h) + \langle \alpha\delta_0 + \beta\delta_1, h\rangle \\ &= \int_\Omega \lambda_1 h\,dx + \alpha h(0) + \beta h(1).\end{aligned} \tag{2.13}$$

Now since $\lambda_1^* \in (H^1(\Omega))^*_+$, it follows that $\lambda_1 \in (H^{-1}(\Omega))_+$ and α, $\beta \geqslant 0$. Hence, we may rewrite Eq. (2.12) as

$$\int_\Omega (-2u_{xx} P^i_{xx} - \lambda_1)\, h\,dx + \alpha h(0) + \beta h(1) = 0 \tag{2.14}$$

and (2.10) as

$$\int_\Omega \lambda_1 (v - a)\,dx + \alpha(v - a(0)) + \beta(v - a(1)) = 0 \tag{2.15}$$

or

$$\int_\Omega \lambda_1 (v - a)\,dx = \alpha(a(0) - v) + \beta(a(1) - v). \tag{2.16}$$

But the right side of (2.16) is nonnegative while the left side is nonpositive. We conclude from (2.15) that

$$\int_\Omega \lambda_1 (v - a)\,dx = 0. \tag{2.17}$$

Similarly, we see that

$$\alpha(a(0) - v) = 0 \qquad \text{and} \qquad \beta(a(1) - v) = 0.$$

From Eq. (2.14) with $h \in H_0^1(\Omega)$, it follows that

$$\lambda_1 = -2u_{xx} P^i_{xx} \tag{2.18}$$

in $H^{-1}(\Omega)$. Now by the regularity results it follows that $u_{xx} P^i_{xx} \in C^0(\bar{\Omega})$ for $i = 1, 2$ and λ_1 is a continuous function. From Eq. (2.18) we see then that

$$\lambda_1(x)(v - a(x)) = 0 \qquad \text{in } \Omega. \tag{2.19}$$

Now if $a(x_0) > v$ for some $x_0 \in \Omega$, we see that there is an open interval I, $x_0 \in I$ such that $\lambda_1 = 0$ in I. From (2.18) it follows that

$$u_{xx}(x)\, P^{(i)}_{xx}(x) = 0 \qquad \text{in } I. \tag{2.20}$$

We consider $i = 1$ and $i = 2$ separately. Let $i = 1$. In this case if $P_{xx} = 0$ on an interval $I_1 \subset I$ then $u_{xx} = 0$ on I_1 with $u_x = c_1$ and $u = c_1 x + c_2$. Hence, we see that on I_1,

$$-a_1(x)\, c_1 + a_2(x)(c_1 x + c_2) = f(x).$$

Otherwise, $P_{xx} = 0$ on an interval $I_2 \subset I$, contradicting (C2). Otherwise, $P_{xx} = 0$ on an interval $I_2 \subset I$ with $P_x = c_3$ and $P = c_3 x + c_4$. Thus, we see that in I_2,

$$-a_1(x)\, c_3 + a_2(x)(c_3 x + c_4) = u(x) - z(x),$$

contradicting (C1).

For the second case $i = 2$ we find the same result as above for $P_{xx} \neq 0$ on I_1. If, however, $P_{xx} = 0$ on I_1, then it follows that

$$-a_{1x} c_3 + a_2(x)(c_3 x + c_4) = \sum_{i=1}^{\omega} (u(x_i, a) - z_i)\, \delta_{x_i}. \tag{2.21}$$

From (2.21) it is clear that $x_i \notin I_1$ for any $i = 1, ..., \omega$ since the left side is in $H^0(I_1)$ and the right side would be in $H^{-(1/2+\varepsilon)}(I_1)$. Thus, we see that

$$(-a_{1x} + a_2(x)\, x)\, c_3 + a_2(x)\, c_4 = 0 \qquad \text{in } I_1$$

and are linearly dependent as functions on I_1. Therefore, we may express

$$a_2(x) = \frac{a_{1x}}{x + c}.$$

3. Penalization and Approximation

To approximate constrained problems, one commonly used method is the so-called penalty method [9]. We discuss the application of this method to the problem

$$\begin{gathered} \text{minimize } J(a) = \| u(a) - z \|^2_{L^2(\Omega)} \\ \text{subject to } a \in H^1(\Omega),\ a \geqslant v,\ \| a \|_{H^1(\Omega)} \leqslant k \end{gathered} \tag{3.1}$$

since the point observation problem is handled in the same manner. We introduce the following functionals on $H^1(\Omega)$,

$$\Psi(a)=\begin{cases}(\|a\|_1^2-k^2)^2, & \|a\|_1>k\\ 0, & \|a\|_1\leqslant k\end{cases} \tag{3.2}$$

and

$$J_\varepsilon(a)=J(a)+\frac{1}{\varepsilon}\Psi(a), \tag{3.3}$$

in order to penalize the norm constraint.

We consider the family of problems for $\varepsilon>0$,

$$\text{minimize } J_\varepsilon(a) \text{ subject to } a\in Q. \tag{3.4}$$

The following is an immediate consequence of the weak lower semicontinuity of Ψ.

PROPOSITION 3.1. *For each $\varepsilon>0$ there exists a solution a_ε of problem* (4.4).

We now investigate the limit behavior of the solutions a_ε as $\varepsilon\to 0$.

LEMMA 3.1. *Let a_ε be a solution of* (3.4). *Then we have the estimate*

$$\|a_\varepsilon\|_1\leqslant(k^2+\varepsilon^{1/2}\|u(v)-z\|_0)^{1/2}.$$

Proof. Since $0<v<k$, we see that the constant function $a\equiv v$ is admissible. Thus, it follows that

$$J_\varepsilon(a_\varepsilon)\leqslant J_\varepsilon(v)=\|u(v)-z\|_{L^2(\Omega)}^2.$$

Accordingly, we have

$$\|u(a_\varepsilon)-z\|_1^2+\frac{1}{\varepsilon}\Psi(a_\varepsilon)\leqslant J(v)$$

and

$$\Psi(a_\varepsilon)\leqslant\varepsilon J(v). \tag{3.5}$$

Now, if $\|a_\varepsilon\|_1\leqslant k$, then the result is immediate. However, if $\|a_\varepsilon\|1>k$, then from (4.5) we see that

$$\|a_\varepsilon\|_1^2-k^2\leqslant\varepsilon^{1/2}(J(v))^{1/2}$$

and the result follows. ▮

We conclude from Lemma 3.1 that the set of solutions $\{a_\varepsilon : \varepsilon > 0\}$ is weakly compact in $H^1(\Omega)$. Hence, we consider a sequence $\{a_{\varepsilon_i}\}_{i=1}^{\infty}$ for $\varepsilon_i \to 0$.

THEOREM 3.1. *Every weak $H^1(\Omega)$ cluster point $\tilde{a}_0$ of the sequence $\{a_{\varepsilon_i}\}_{i=1}^{\infty}$ as $\varepsilon_i \to 0$ is a solution of* (4.1).

Proof. Let $\tilde{a}_0$ be a weak $H^1(\Omega)$ cluster point of the sequence $\{a_{\varepsilon_i}\}_{i=1}^{\infty}$. Hence, there is a subsequence $\{a_{\varepsilon ij}\}_{j=1}^{\infty}$ such that $a_{\varepsilon_{ij}} \to \tilde{a}_0$ weakly in $H^1(\Omega)$. From the estimate (3.5) and the weak lower semicontinuity of Ψ in $H^1(\Omega)$, it follows that

$$0 = \lim \varepsilon_{ij} J(v) \geqslant \underline{\lim}\, \Psi(a_{\varepsilon_{ij}}) \geqslant \Psi(\tilde{a}_0).$$

Thus, we conclude that $\bar{a}_0$ satisfies the norm constraint in (4.1). The pointwise constraint follows from the fact that $H^1(\Omega)$ is compactly imbedded in $C^0(\bar{\Omega})$.

Now we observe that if a_0 is a solution of (4.1) then

$$J(a_0) = J_\varepsilon(a_0) \geqslant J_\varepsilon(a_\varepsilon) = J(a_\varepsilon) + \frac{1}{\varepsilon}\, \Psi(a_\varepsilon). \tag{3.6}$$

Since $a_{\varepsilon_{ij}} \to \tilde{a}_0$ weakly in $H^1(\Omega)$, we see that

$$J(a_{\varepsilon_{ij}}) = \| u(a_{\varepsilon_{ij}}) - z \|^2_{L^2(\Omega)} \to \| u(\tilde{a}_0) - z \|^2_{L^2(\Omega)} = J(\tilde{a}_0).$$

Hence, given a $\delta > 0$ there exists $N > 0$ such that if $j \geqslant N$ then

$$J(a_{\varepsilon_{ij}}) \geqslant J(\tilde{a}_0) - \delta.$$

Since $\tilde{a}_0$ is admissible, we have

$$J(a_{\varepsilon_{ij}}) \geqslant J(a_0) - \delta.$$

Substituting into (3.6) we find that for any $j \geqslant N$,

$$\delta \geqslant \frac{1}{\varepsilon_{ij}}\, \Psi(a_{\varepsilon_{ij}}).$$

Since δ is arbitrary, we conclude that

$$\lim_{j \to \infty} \frac{1}{\varepsilon_{ij}}\, \Psi(a_{\varepsilon_{ij}}) = 0.$$

Returning to (3.6), we see that

$$J(a_0) \geqslant \lim_{j \to \infty} J(a_{\varepsilon_{ij}}) + \lim_{j \to \infty} \frac{1}{\varepsilon_{ij}}\, \Psi(a_{\varepsilon_{ij}})$$

and

$$J(z_0) \geqslant J(\tilde{a}_0).$$

Since $\tilde{a}_0$ belongs to the admissible set in (4.1) we see that $\tilde{a}_0$ is a solution of (3.1). ∎

Using arguments similar to those in Section 2, we may deduce regularity properties of solutions of (4.4).

THEOREM 3.2. *Suppose that assumptions* (A1)–(A3) *hold. Then for any solution,* a_ε *of* (3.4) *for* $\varepsilon > 0$ *belongs to* $H^3(\Omega)$.

We now turn to the approximation of penalized problems by finite-dimensional problems. In particular we determine limit behavior as penalization and the dimension of the approximating spaces become larger. To formulate the finite-dimensional estimation problem we recall Eq. (1.4),

$$L(u, v, a) = (f, v),$$

where $L(\cdot, \cdot, a)$ is given by (1.1). To approximate a, we consider a concrete subspace $\Phi^M \subset H^1(\Omega)$ of linear splines with knots at k/M, $k = 0, 1, ..., M$, with basis elements

$$\phi_k(x) = \begin{cases} Mx - (k-1), & x \in ((k-1)/M, k/M) \\ -Mx + (k+1), & x \in (k/M, (k+1)/M) \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we look for a^M in the form $a^M(x) = \sum_{k=1}^{M+1} \alpha_k \phi_k(x)$.

Denote the linear interpolation operator by L^M, where $L^M: H^1(\Omega) \to \Phi^M$ (cf. [10]).

Remark 3.1. We recall several properties of L^M.

(i) If $a \in H^1(\Omega)$, $a \geqslant v$, then $L^M a \geqslant v$.

(ii) If $a \in H^1(\Omega)$, then $\| a - L^M a \|_1 \leqslant C \| a \|_1$.

(iii) If $a \in H^2(\Omega)$, then $\| a - L^M a \|_1 \leqslant (C/M) \| a \|_2$, where C represents a constant independent of M.

We now formulate a finite-dimensional problem to approximate (3.4):

$$\begin{gathered} \text{minimize } J_\varepsilon^N(a) = \| u^N(a) - z \|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \Psi(a) \\ \text{subject to } a \in Q^M, \end{gathered} \tag{3.7}$$

where $Q^M = \{a \in \Phi^M : a \geqslant v\}$. Observe from Remark 3.1(i) and the definition (1.1) of Q that

$$L^M Q \subseteq Q^M \subseteq Q. \tag{3.8}$$

The existence of a solution $a_\varepsilon^{N,M}$ to (3.7) may be easily proved using well-known arguments.

THEOREM 3.3. *There exists a solution $a_\varepsilon^{N,M}$ to problem* (3.7).

We now establish limit behavior of $\{a_\varepsilon^{N,M}\}$ for $\varepsilon > 0$ fixed.

THEOREM 3.4. *Let $\varepsilon > 0$ be fixed and let $(N_i, M_i) \to (\infty, \infty)$ as $i \to \infty$. Then $\{a_\varepsilon^{N_i, M_i}\}$ is bounded in $H^1(\Omega)$ and every weak $H^1(\Omega)$ cluster point is a solution* (3.4).

Proof. Boundedness of the sequence $\{a_\varepsilon^{N_i, M_i}\}$ may be established using the facts that v belongs to Q^{N_i} for every i, inequality (3.6) holds, and $u^{M_i}(v) \to u(v)$ in $L^2(\Omega)$. Hence, there is a subsequence $\{a_\varepsilon^{N_{ij}, M_{ij}}\}$ such that

$$a_\varepsilon^{N_{ij}, M_{ij}} \to \tilde{a}_\varepsilon$$

weakly in $H^1(\Omega)$. To show that $\tilde{a}_\varepsilon$ is a solution of (3.7) is similar to the proof of Theorem 3.2.

We now use regularity results of Section 2 to obtain a convergence theorem for the $a_\varepsilon^{N,M}$. Recall from Theorem 2.5 that $\bar{a}_0 \in H^{2-p}(\Omega)$ for $1 > p > \frac{1}{2}$. We see that from (3.4),

$$\| L^M \bar{a}_0 - \bar{a}_0 \|_{H^1(\Omega)} \leqslant C(1/M)^{1-p} \, \| \bar{a}_0 \|_{H^{2-p}(\Omega)}. \tag{3.9}$$

Thus, we have

$$\begin{aligned} \| L^M \bar{a}_0 \|_{H^1(\Omega)} &\leqslant \| \bar{a}_0 \|_{H^1(\Omega)} + C(1/M)^{(1-p)} \, \| \bar{a}_0 \|_{H^{2-p}(\Omega)} \\ &\leqslant (k + C(1/M)^{(1-p)}(\| \bar{a}_0 \|_{H^{2-p}(\Omega)})) \end{aligned} \tag{3.10}$$

and

$$\psi(L^M \bar{a}_0) \leqslant 2(k + C \, \| \bar{a}_0 \|_{H^{2-p}(\Omega)}) \, C(1/M)^{1-p} \, \| \bar{a}_0 \|_{H^{2-p}(\Omega)}.$$

Now set $\varepsilon_M = M^{\alpha(p-1)}$, $0 < \alpha < 1$. Then we see that for $\bar{a}_0$, a solution of (4.3),

$$\begin{aligned} J_{\varepsilon_M}^N(a_M^{N,M}) &= J^N(a_{\varepsilon_M}^{N,M}) + \frac{1}{\varepsilon_M} \psi(a_{\varepsilon_M}^{N,M}) \\ &\leqslant J_{\varepsilon_M}^N(L^M \bar{a}_0) \\ &\leqslant J^N(L^M \bar{a}_0) + M^{\alpha(1-p)} \Psi(L^M \bar{a}_0). \end{aligned} \tag{3.11}$$

From the boundedness of $a_{\varepsilon_M}^{N,M}$ there is a sequence $a_{\varepsilon_{M_i}}^{N,M}$ such that $a_{\varepsilon_{M_i}}^{N_i,M_i} \to \tilde{a}_0$ weakly in $H^1(\Omega)$. It follows that $\tilde{a}_0$ is admissible for (3.1). Moreover, from (3.11) and Theorem 1.5, we see that in the limit

$$J(\tilde{a}_0) \leqslant J(\bar{a}_0)$$

and $\tilde{a}_0$ must be a solution of (3.1). ∎

THEOREM 3.5. *Assume* (C1)–(C3) *and let* $\varepsilon_M = M^{\alpha(1-p)}$, *where* $1 > p > \frac{1}{2}$ *and* $1 > \alpha > 0$. *Then any weak* $H^1(\Omega)$ *cluster point* $\tilde{a}_0$ *of the solutions* $a_{\varepsilon_{M_i}}^{N_i,M_i}$ *is a solution of* (3.1).

4. NUMERICAL EXPERIMENTS

In this section we describe some numerical experiments we have conducted to solve (3.7). To this end we rewrite the problem as matrix equations. Given Eq. (1.4) we have

$$L(u^N, B_k; a^M) = (f, B_k), \qquad k = 1, ..., l(N), \tag{4.1}$$

where $u^N = \sum_{i=1}^{l(N)} c_i B_i$, $S^N = \text{span}\{B_i: i = 1,, ..., l(N)\}$, and $a^M = \sum_{j=1}^{M+1} \alpha_j \phi_j$. In all our computations the B_i are cubic B-splines that have been adjusted so that $B_i \in H_0^2(\Omega)$. Equations (4.1) may be written as

$$\sum_{i=1}^{l(N)} c_i \left(\sum_{j=1}^{M+1} \alpha_j L(B_i, B_k, \phi_j) \right) = (f, B_k).$$

Define the matrices G_j and G by

$$(G_j)_{ik} = L(B_i, B_k, \phi_j) \qquad \text{and} \qquad G(\tilde{\alpha}) = \sum_{j=1}^{M+1} \alpha_j G_j$$

and vectors $\tilde{\alpha}$, $\tilde{c}$, and $\tilde{f}$ by

$$(\tilde{\alpha})_j = \alpha_j, j = 1, ..., M+1, \qquad (\tilde{c})_k = c_k, \qquad (\tilde{f})_k = (f, B_k), k = 1, ..., l(N).$$

In matrix form (4.1) is given by

$$G\tilde{c} = f, \tag{4.2}$$

where G and $\tilde{c}$ both depend on α.

As for the functional J_ε^N, if we define the matrices $G^{(0)}$ and G^ϕ by

$$(G^{(0)})_{ij} = \int_\Omega B_i B_j \qquad \text{and} \qquad (G^\phi)_{ij} = \int_\Omega (\phi_{ix}\phi_{jx} + \phi_i \phi_j),$$

then we have

$$J_\varepsilon^N(\alpha) = c^T G^{(0)} c - 2c^T\delta + c_z + \frac{1}{\varepsilon}\,\Psi^N(\alpha), \tag{4.3}$$

where $(\delta)_i = \int_\Omega B_i z$, $c_z = \int z^2$, and

$$\Psi^N(\alpha) = \begin{cases} (\alpha^T G^\phi \alpha - k^2)^2, & \alpha^T G^\phi \alpha > k^2 \\ 0, & \alpha^T G^\phi \alpha \leqslant k^2. \end{cases} \tag{4.4}$$

Thus, the optimization problem is given as

$$\begin{aligned} &\text{minimize } J_\varepsilon^N(\alpha) \\ &\text{subject to } \alpha_i \geqslant v,\ i = 1, ..., \end{aligned} \tag{4.5}$$

We use a Newton–Raphson method as a minimization technique where we compute the derivatives of $J_\varepsilon^N(\cdot)$ by

$$D_i J_\varepsilon^N(\alpha) = 2(G\tilde{c} - \delta)^T\,(D_i\tilde{c}) + \frac{1}{\varepsilon}\,D_i \Psi^N(\alpha) \tag{4.6}$$

$$D_{ik}^{(2)} J_\varepsilon^N(\alpha) = 2(D_k\tilde{c})^T G(D_i\tilde{c}) + 2(G\tilde{c} - \delta)^T\,(D_{ik}\tilde{c}) + \frac{1}{\varepsilon}\,D_{ik}\Psi^N(\alpha), \tag{4.7}$$

where D_i and $D_{ik}^{(2)}$ are the partial derivatives with respect to α_i and α_i, α_k, respectively. We may compute $D_i\tilde{c}$ and $D_{ik}\tilde{c}$ by differentiating (4.2) to find

$$G(\alpha)(D_i\tilde{c}) = -G_i\tilde{c} \tag{4.8}$$

and

$$G(\alpha)(D_{ik}\tilde{c}) = -G_k(D_i\tilde{c}) - G_i(D_k\tilde{c}). \tag{4.9}$$

Hence, we generate a sequence of $\alpha^l \in \mathbb{R}^{M+1}$ by the iteratin formula

$$\alpha^{(l+1)} = \alpha^{(l)} - (D^{(2)}J(\alpha^{(l)}))^{-1}\,DJ(\alpha^{(l)}). \tag{4.10}$$

As a test problem we consider the example

$$\begin{gathered} (au_{xx})_{xx} = f \qquad \text{in } (0, 1) \\ u(0) = u(1) = u_x(0) = u_x(1) = 0. \end{gathered}$$

Observation is assumed to be over the entire interval (0, 1) and is

$$z(x) = 4x^2(1-x)^2.$$

We seek to recover

$$a_0(x) = \tfrac{3}{2} + \cos(2\pi x)$$

given the function

$$f(x) = (az_{xx})_{xx}.$$

We consider the case in which $l(N) = 9$ and $M + 1 = 9$ and start the iteration with ${}_i^{(0)} = 1.5$ for $i = 1, ..., 9$. Set $p = 1/\varepsilon$ and $k = (\frac{3}{2} + \pi/8 + 1/8\pi)^{1/2}$.

$x =$	0.0	0.2	0.5	0.8	1.0
a_0 (exact)	2.50	1.81	0.5	1.81	2.50
$p = 0.0$					
	2.38	1.99	0.5	1.99	2.38
L^2 error $= 0.17$	Number of iterations $= 6$				
$x =$	0.0	0.2	0.5	0.8	1.0
$p = 10^{-8}$					
	2.40	1.73	0.5	1.7	2.4
L^2 error $= 0.16$	Number of iterations $= 6$				
$x =$	0.0	0.2	0.5	0.8	1.0
$p = 10^{-6}$					
	2.4	1.71	0.5	1.71	2.4
L^2 error $= 0.09$	Number of iterations $= 5$				
$x =$	0.0	0.2	0.5	0.8	1.0
$p = 10^{-5}$					
	2.58	1.79	0.5	1.81	2.57
L^2 error $= 0.04$	Number of iterations $= 8$				
$x =$	0.0	0.2	0.5	0.8	1.0
$p = 10^{-3}$					
	2.53	1.79	0.5	1.84	2.52
L^2 error $= 0.03$	Number of iterations $= 7$				
$x =$	0.0	0.2	0.5	0.8	1.0
$p = 10^{-2}$					
	2.41	1.84	0.5	1.88	2.40
L^2 error $= 0.17$	Number of iterations $= 4$ Unstable thereafter				

$p \geqslant 10^{-1}$ Unstable

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